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BENGT ROSÉN

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LIMIT THEOREMS FOR SUMS OF RANDOM VARIABLES

INAUGURAL DISSERTATION

BY

BENGT ROSÉN

FIL. LIC., GÄSTR.-HÄLS.

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In submitting this thesis, I want to thank my teachers of mathematics at the University of Uppsala.

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Uppsala, October 1964.

Bengt Rosén

Papers summarized in the dissertation

A. Rosén, B., On the asymptotic distribution of sums of independent identically distributed random variables. *Arkiv för matematik*, B 4 (1961), pp. 323–332.

B. Rosén, B., On the probabilities that a random walk is negative. *Arkiv för matematik*, B 5 (1964), pp. 317–325.

C. Rosén, B., Limit theorems for sampling from finite populations. *Arkiv för matematik*, B 5 (1964), pp. 383–424.

Limit theorems for sums of random variables

Numerous evolutions depending on chance can be described mathematically by a sequence

$$S_1, S_2, \ldots, S_n \ldots \tag{1}$$

of sums of successive random increments, i.e. $S_n = X_1 + X_2 + \ldots + X_n$, $n = 1, 2, \ldots$, where the X's are random variables.

Particularly important and thoroughly treated is the case when the increments are assumed to be independent and equally distributed. The process (1) is then fully determined by the common distribution $F(x)$ of the increments, i.e.

$$F(x) = P(X_1 \leqslant x); \quad -\infty < x < \infty.$$

Note. Throughout this work P stands for probability, EX and $\sigma^2(X)$ denote respectively the mean and the variance of the random variable X.

Fundamental results on the asymptotic behaviour of S_n are the laws of large numbers and the central limit theorem. The latter theorem states that if $0 < \sigma^2(X_1) < \infty$, then

$$P(S_n \leqslant nEX_1 + \alpha\sigma(X_1)\sqrt{n}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{x^2}{2}}\, dx + R_n(\alpha), \tag{2}$$

where the remainder $R_n(\alpha)$ tends to zero uniformly in α, when $n \to \infty$.

Usually of greater mathematical complexity are problems concerning the behaviour of random quantities which depend not only on S_n but simultaneously on $S_1, S_2, \ldots, S_n$. An example of such a quantity is N_n, the number of positive sums among $S_1, S_2, \ldots, S_n$.

The works of E. Sparre Andersen (see [1] where also ref. to earlier works can be found) and F. Spitzer [4] were instrumental in the substantial advancement of the theory of the behaviour of N_n and other similar random variables. As a typical example of the results they obtained, we list the following formula, which is essentially due to Andersen

$$\sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} \lambda^m P(N_n = m) = \exp\left[\sum_{n=1}^{\infty} \frac{t^n}{n} (\lambda^n P(S_n > 0) + P(S_n \leqslant 0))\right]. \tag{3}$$

By the formula (3) the probabilities desired, namely $P(N_n = m)$, $m = 0, 1, 2, \ldots, n$, $n = 1, 2, \ldots$, are determined in terms of the probabilities

$$P(S_n<0), \quad P(S_n=0), \quad P(S_n>0), \qquad n=1,2,\ldots \tag{4}$$

and the asymptotic behaviour of N_n can be deduced from the asymptotic behaviour of the quantities in (4).

A main concern in papers [A] and [B] is to characterize the asymptotic behaviour of the probabilities in (4) in terms of $F(x)$. The problems considered are inspired by questions raised in some of Spitzer's papers. The methods used are to a large extent based on Fourier analysis.

So far we have considered sums of independent random variables. Paper [C] is concerned with non-independent random variables, namely, samples from finite populations. By a finite (one-dimensional) population of size N, denoted $\pi=\{a_1, a_2, \ldots, a_N\}$, we mean a set of N real numbers. A random sample $X_1, X_2, \ldots, X_n$ from π is obtained by drawing, without replacement, $n\,(n\leqslant N)$ elements at random from π.

The result (2) is a typical probability theorem in that it is a limit theorem. The main application of (2) is obtained by disregarding the remainder term to get an approximation of the distribution of S_n. For sampling from finite populations, approximations to exact solutions can also be obtained by considering a limit procedure. A fruitful limit procedure, studied earlier by many authors (for ref. see [C]), is obtained by the following scheme:

$$\begin{array}{lll}
X_{11}, X_{12}, \ldots, X_{1n_1} & \text{is a random sample from} & \pi_1 \\
X_{21}, X_{22}, \ldots, X_{2n_2} & \text{is a random sample from} & \pi_2 \\
\vdots \quad \vdots \quad \vdots & & \vdots \\
X_{k1}, X_{k2}, \ldots, X_{kn_k} & \text{is a random sample from} & \pi_k \\
\vdots \quad \vdots \quad \vdots & & \vdots
\end{array}$$

For $k=1, 2, \ldots$, let $\varphi_k(X_{k1}, X_{k2}, \ldots, X_{kn_k})$ be a random variable defined in terms of the sample $X_{k1}, X_{k2}, \ldots, X_{kn_k}$. An example is furnished by the sample sum

$$S_{n_k}^{(k)} = X_{k1} + X_{k2} + \ldots + X_{kn_k}.$$

Paper [C] is devoted to a study of limit problems connected with some types of functions φ_k, especially when φ_k is the sample sum. The conditions for applications of the results can be indicated as follows. The distribution of $\varphi_k(X_{k1}, X_{k2}, \ldots, X_{kn_k})$ can be approximated by the corresponding limit distribution if φ_k is based on a "fairly large" sample from a "large" population π_k which has a "regular" shape. We give a brief review of the main results in [C].

In chapter 2 we prove analogues of the laws of large numbers and the law of the iterated logarithm.

In chapter 4 we consider the distribution of sample sums. A precise formulation of the following heuristic statement is presented. If π_k does not contain discordant

elements, $S_{n_k}^{(k)}$ has approximately a normal distribution; however, if π_k contains discordant elements, then the distribution of $S_{n_k}^{(k)}$ differs significantly from a normal distribution.

We also show that the process

$$S_1^{(k)}, S_2^{(k)}, \ldots, S_{n_k}^{(k)}$$

after proper normalization can be described approximately by a "tied down" Brownian motion process. This result yields a counterpart of the so-called invariance principle, see [3].

In chapter 5 we consider empirical distributions and empirical fractiles. The results can be roughly characterized as follows. The main features from the case of independent observations carry, under suitable conditions, over to the case of sampling from finite populations. The main difference between the two cases lies in the normalizing factors. In particular we prove analogues of Kolmogorov's theorem on the asymptotic distribution of the discrepancy between the empirical and the the theoretical distributions, and a result concerning asymptotic normality of empirical fractiles, see [2] p. 369.

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